

Quantum continuous measurements, dynamical role of information and restricted path integrals

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Abstract

The restricted-path-integral (RPI) theory of continuous quantum measurements including the evolution of the measured systems and phenomenon of decoherence is reviewed. The measured system is considered as an open quantum system but without usage of any model of the measurement (of the measuring medium or the system's environment). The propagator of a measured system (conditioned by the measurement readout) is presented by RPI. In the important special case of monitoring an observable the propagator and the system's wave function satisfy Schrödinger equation with a complex Hamiltonian (depending on the measurement readout). Going over to the non-selective description of the measurement leads to the Lindblad master equation. In case of non-minimally disturbing measurements this gives theory of dissipative systems avoiding difficulties of other approaches. The whole theory is deduced from first principles of quantum mechanics. This proves that quantum mechanics includes theory of measurements and is therefore conceptually closed.

1 Introduction

Theory of quantum measurements is one of the most interesting and in a sense mysterious topics in quantum mechanics. In last decades essential progress in understanding quantum measurements was connected with the

concept of decoherence. The phenomenon of decoherence consists in loss of quantum phases in the superposition existed before the measurement, see [1] for a review. Decoherence transforms an initial pure state (presented by a state vector, or wave function) to a mixed state (presented by a density matrix). Continuous quantum measurement (gradual decoherence) may be presented by a time-dependent density matrix. In Markowian approximation this density matrix satisfies a differential equation called master equation. Generic form of master equations was found by Lindblad [2].

It is important that a quantum open system may be considered to be continuously measured by its environment even if the latter is not constructed for measurement. Therefore, theory of continuously measured systems is in fact theory of open systems. We give here a short review of theory of continuous quantum measurements based on restricted path integrals (RPI).

Decoherence is caused by entanglement (quantum correlation) between the measured system and its environment (measuring medium or reservoir). The density matrix presents the result of decoherence in a non-selective way when the measurement readout (final state of the environment) is not known. The same physical process of measurement may be presented in the selective way, by a state vector conditioned by the measurement readout.

In case of a continuous measurement (gradual decoherence) the time-dependent readout-conditioned state vector may be presented by RPI which in Markowian approximation satisfies Schrödinger equation with a non-Hermitian Hamiltonian depending on the measurement readout, see [3] for a review.

RPI approach to continuous measurement follows from first principles of quantum mechanics in the form of Feynman's path integrals but may also be derived by ordinary quantum-mechanical methods. The approach is model-independent: RPI depends only on the information supplied by the measurement but not on the concrete measuring medium. Some features of RPI point out that the evolution of a continuously measured (decohering) system is a fundamental type of evolution. Particularly, RPI approach reveals the dynamical role of the information escaping the system.

RPI differs from the usual Feynman path integral by a weight functional in the integrand. Positive weight functionals describe continuous measurements which result in minimal disturbance of the measured system's state (given the information supplied by the measurement). Those weight functionals which contain also phase shifts, determine non-minimally disturbing measurements leading to dissipation of the measured system.

2 Continuous measurements and RPI

Any measurement of a quantum system results from interacting this system with its environment. The direct way to describe the evolution of the measured system is to find the time-dependent density matrix of the closed system consisting of the system of interest and its environment and then trace out all degrees of freedom of the environment. The resulting reduced density matrix describes the measured system as an open system i.e. with the influence of the environment taken into account. This description is non-selective since all possible states of the environment (all possible measurement readouts) are accounted in it, with no selection of one of them. Examples of continuous measurements considered in this way may be found in [4].

It is however possible to derive the evolution law of a continuously measured system phenomenologically, making use of no explicit model of the environment and interaction with it. We shall show how this may be done with the help of restricted path integrals (RPI), see [3] for the review of the method. The resulting description will be selective, expressed in terms of a state vector (wave function) satisfying Schrödinger equation with the complex potential depending on the measurement readout. A non-selective description in the form of a master equation for the density matrix may be obtained then by integration over all possible measurement readouts.

According to Feynman's formulation of quantum mechanics, the propagator (probability amplitude for transition from one point to another) of a *closed* quantum system may be expressed in the form of a path integral

$$U_t(q'', q') = \int d[p] \int_{q'}^{q''} d[q] e^{\frac{i}{\hbar} \int_0^t (p\dot{q} - H(p, q, t)) dt} \quad (1)$$

where the so-called *phase-space, or Hamiltonian, representation of the path integral* is used. Integration must be performed in all paths $[p]$ in the momentum space and in those paths $[q]$ in the configuration space which have the given end points. The observables p, q may be multidimensional.

Details of the definition are not important for us here. The only thing we need is that the propagator $U_t(q'', q')$ and the corresponding time-dependent wave function $\psi_t(q'')$ satisfy Schrödinger equation. If we introduce the evolution operator U_t as an integral operator with the kernel $U_t(q'', q')$, then the evolution of the system may be presented, correspondingly by the state

vector or density matrix, as follows:

$$|\psi_t\rangle = U_t |\psi_0\rangle, \quad \rho_t = U_t \rho_0 U_t^\dagger. \quad (2)$$

The ideology which was developed by Feynman as a background of the path-integral approach is following. The probability amplitude for the system to evolve along the given path is equal to the integrand in (1) (imaginary exponential of the action in units of the Plank constant). However it is in principle impossible to know for a closed system what path is chosen by it. Therefore, we have to sum up the amplitudes corresponding to all paths, hence Eq. (1) for the total probability amplitude (propagator).

If the system is continuously measured, this ideology directly leads to RPI. Indeed, let a continuous measurement is performed on the system during the time interval $[0, t]$. Then the measurement readout supplies some information about the path chosen by the system in its evolution. If α is the set of paths compatible with this information, then the path integral has to be restricted onto the set α . The set α is called *quantum corridor*.

Instead, we may integrate over all paths but with the corresponding weight functional in the integrand. This weight functional $w_\alpha[p, q]$ has to be equal to unity for the paths $[p, q]$ belonging to α and zero otherwise. Generalizing, we may describe the measurement by any smooth functional $w_\alpha[p, q]$. Although no set of paths can present this situation adequately, we may in this case speak of the ‘quantum corridor with indistinct, fuzzy boundaries’. In fact, this is a more realistic description, since real measurements give information of just this type. α will be interpreted in this case either as a measurement readout or as a quantum corridor presented by $w_\alpha[p, q]$.

As a result, we have the following propagator for the system undergoing a continuous measurement resulting in the measurement readout α :

$$U_t^\alpha(q'', q') = \int d[p] \int_{q'}^{q''} d[q] w_\alpha[p, q] e^{\frac{i}{\hbar} \int_0^t (pq - H(p, q, t)) dt}. \quad (3)$$

The evolution of the system is then presented as follows:

$$|\psi_t^\alpha\rangle = U_t^\alpha |\psi_0\rangle, \quad \rho_t^\alpha = U_t^\alpha \rho_0 (U_t^\alpha)^\dagger \quad (4)$$

The system undergoing the measurement (therefore decohering) may thus be presented by wave functions, or state vectors, but these wave functions are conditioned by the measurement readouts α . Physically a measurement

readout is nothing else than a state of the environment, but here measurement readouts are presented by weight functionals $w_\alpha[p, q]$ i.e. expressed in terms of paths of the measured system. This is often advantageous.

Instead of a single propagator and a single unitary evolution operator for a closed system, a continuously measured system is presented by the set of *partial propagators* (3) and *partial evolution operators* U_t^α , one for each measurement readout α . These operators are not unitary, so that the state vectors and density matrices (4) are not normalized. Instead, the square norm of the state vector or trace of the density matrix gives the probability density of the corresponding measurement readout:

$$P(\alpha) = \|\psi_t^\alpha\|^2 = \text{tr } \rho_t^\alpha \quad (5)$$

The sum of the *partial density matrices* corresponding to all possible measurement readouts gives the *total density matrix*

$$\rho_t = \int d\alpha \rho_t^\alpha = \int d\alpha U_t^\alpha \rho_0 (U_t^\alpha)^\dagger \quad (6)$$

presenting the same measurement non-selectively. The total density matrix ρ_t should be normalized. This is the case if the *generalized unitarity condition*

$$\int d\alpha (U_T^\alpha)^\dagger U_T^\alpha = \mathbf{1} \quad (7)$$

is fulfilled. Eq. (7) is a condition on the weight functionals w_α and the measure $d\alpha$ which provides consistency of the definitions.

Of course, one may integrate over some subset of the measurement readouts, providing a partially non-selective description of the measurement. In practice only various partially non-selective descriptions are realistic because the measurement readout α never can be known precisely.

3 Monitoring an observable

Let us consider a special case of continuous measurements, monitoring an observable. An evident example is monitoring the coordinate q . The measurement readout is then expressed by function $a(t')$, $t' \in [0, t]$, and interpreted as the statement that at time t' the coordinate q has the value which differs from $a(t')$ not more than by the entity Δa characterizing the measurement

accuracy. The same may be expressed as follows: the system evolves along some path $[q]$ (in the coordinate space) which lies in the corridor α of the width Δa with the middle line $[a] = \{a(t') | 0 \leq t' \leq t\}$. Such a measurement is described by RPI taken over the quantum corridor α .

In generic case any observable $A(p, q)$ is monitored instead of the coordinate. The measurement readout expressed by the curve $[a] = \{a(t') | 0 \leq t' \leq t\}$ means that the value $A(t') = A(p(t'), q(t'))$ of the observable $A(p, q)$ at time t' differs from $a(t')$ not more than by Δa (the precision, or resolution, of the measurement). RPI must be taken over the set α of paths $[p, q]$ determined by these conditions.

Of course, a more realistic description of the measurement is given by the corridor with fuzzy boundaries. It is expressed by a weight functional $w_\alpha[q]$ for monitoring the coordinate and by $w_\alpha[p, q]$ in the general case. The functional $w_\alpha[p, q]$ for monitoring $A(p, q)$ has to be approximately equal to unity if the curve $A(t') = A(p(t'), q(t'))$, $t' \in [0, t]$, is close to $a(t')$ and approximately equal to zero if these curves are far from each other.

The scale of closeness is Δa , but the definition of the ‘distance’ between curves may depend on the concrete type of the measurement to be described. It is often reasonable to present monitoring by the Gaussian functional

$$w_{[a]}[p, q] = \exp \left(-\kappa \int_0^t [A(t') - a(t')]^2 dt' \right) \quad (8)$$

where κ determines the strength, or resolution, of the measurement.

In principle κ may depend on time, then it has to be in the integrand, but for permanent conditions of the monitoring $\kappa = \text{const}$. If the interval t of measurement is fixed, the strength of the measurement κ is connected with the ‘width of the corridor’ Δa by the equation $\kappa = 1/(t \Delta a^2)$, so that Δa is the mean square deviation of the curve $A(t)$ from the middle line of the corridor $a(t)$. Therefore, for a constant strength of the measurement, Δa decreases with time inversely proportional to \sqrt{t} .

The first, purely mathematical, reason to choose a Gaussian weight functional is that this leads (in case of a quadratic Hamiltonian) to Gaussian path integrals which may be precisely evaluated. There is however a more deep physical reason. As is shown in the framework of a special model [3, Chapter 8] and is believed to be valid generally, a Gaussian weight functional appears each time if the continuous measurement consists of a large number

of very weak short measurements. This may be considered as a quantum version of the Central Limiting Theorem from probability theory.

If the Gaussian weight functional (8) is accepted for monitoring, then RPI (3) takes the form

$$U_t^{[a]}(q'', q') = \int d[p] d[q] \exp \left\{ \frac{i}{\hbar} \int_0^t (p\dot{q} - H(p, q, t)) dt - \kappa \int_0^T (A(p, q, t) - a(t))^2 dt \right\} \quad (9)$$

This RPI is equal to Feynman path integral (1) but with the Hamiltonian

$$H_{[a]}(p, q, t) = H(p, q, t) - i\kappa\hbar (A(p, q, t) - a(t))^2 \quad (10)$$

containing (in comparizon with the initial Hamiltonian) an additional imaginary term. The propagator (9) and the corresponding conditioned wave function (state vector) $|\psi_t^{[a]}\rangle$ satisfy the effective Schrödinger equation

$$\frac{\partial}{\partial t} |\psi_t^{[a]}\rangle = \left[-\frac{i}{\hbar} \hat{H} - \kappa (\hat{A} - a(t))^2 \right] |\psi_t^{[a]}\rangle \quad (11)$$

The transition to the non-selective description of the measurement is performed by summing up the partial density matrices over all possible measurement readouts. In case of monitoring, this means integrating $\rho_t^{[a]}$ over all curves $[a]$. The resulting total density matrix may be shown to satisfy the following equation (which is a special case of Lindblad equation):

$$\frac{\partial}{\partial t} \rho_t = -\frac{i}{\hbar} [\hat{H}, \rho_t] - \frac{1}{2} \kappa [\hat{A}, [\hat{A}, \rho_t]] \quad (12)$$

4 Non-minimally disturbing monitoring

We shall consider now a more general *non-minimally disturbing monitoring* [3, Sect. 5.2.3]. It is described by a more general weight functional than (8). The functional (8) damps out those paths which are not compatible with the information given by the monitoring. This leads to such a disturbance of the measured system's state which is unavoidable for the measurement supplying the given information. Besides this, some additional (non-minimal) disturbance may be performed during the measurement.

For example when the coordinate is measured, the momentum is necessarily disturbed, but, besides this, the measurement of the coordinate may be accompanied by additional disturbance of the coordinate.

A non-minimally disturbing monitoring of observable $A(p, q)$ may be described by a Gaussian weight functional similar to (8) but with an additional imaginary term in the exponent. If we assume that this additional term is linear in the measurement readout $a(t)$, then the weight functional is

$$w_{[a]}[p, q] = \exp \left\{ \int_0^t dt' \left[-\kappa (A - a(t'))^2 - \frac{i}{\hbar} (\lambda a(t') B + C) \right] \right\} \quad (13)$$

where $A = A(p, q)$, $B = B(p, q)$, $C = C(p, q)$ are arbitrary observables. This type of measurement leads to dissipation of the measured system [5].

This may be shown as follows. If the partial evolution operator $U_t^{[a]}$ is RPI with weight functional (13), then the partial and total density matrices

$$\rho_t^{[a]} = U_t^{[a]} \rho_0 \left(U_t^{[a]} \right)^\dagger, \quad \rho_t = \int d[a] \rho_t^{[a]} \quad (14)$$

describe the measurement, correspondingly, selectively and non-selectively. It turns out [5] that the total density matrix satisfies the master equation

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [\hat{H} + \hat{C}, \rho] - \frac{\kappa}{2} [\hat{A}, [\hat{A}, \rho]] - \frac{\lambda^2}{8\kappa\hbar^2} [\hat{B}, [\hat{B}, \rho]] - \frac{i\lambda}{2\hbar} [\hat{B}, [\hat{A}, \rho]_+]. \quad (15)$$

(we omit index t in ρ_t). With the notation $\hat{l} = \hat{A} - i\frac{\lambda}{2\kappa\hbar}\hat{B}$, we have

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} \left[\hat{H} + \hat{C} - i\frac{\kappa\hbar}{4} (\hat{l}^{\dagger 2} - \hat{l}^2), \rho \right] - \frac{\kappa}{2} (\hat{l}^\dagger \hat{l} \rho - 2\hat{l} \rho \hat{l}^\dagger + \rho \hat{l} \hat{l}^\dagger) \quad (16)$$

This equation is of Lindblad form [2] with a single Lindblad operator \hat{l} . The original Hamiltonian is renormalized by the measurement procedure.¹

From the physical point of view the measured system turns out to be dissipative. For example, if \hat{H} is a Hamiltonian of the harmonic oscillator, $\hat{A} = \hat{p}$, $\hat{B} = \hat{q}$ and $\hat{C} = 0$ (this means that the oscillator's momentum is monitored and non-minimally disturbed), then the resulting master equation

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [\hat{H}, \rho] - \frac{\kappa}{2} [\hat{p}, [\hat{p}, \rho]] - \frac{\lambda^2 \omega^2}{8\kappa\hbar^2} [\hat{q}, [\hat{q}, \rho]] - \frac{i\lambda\omega}{2\hbar} [\hat{q}, [\hat{p}, \rho]_+] \quad (17)$$

¹The Lindblad master equation of generic form, i.e. with a number of Lindblad operators, will result if the corresponding number of observables undergo non-minimally disturbing monitoring.

is the well known master equation for a *Brownian motion* of the harmonic oscillator [6]. The Brownian motion of the oscillator is thus interpreted as the *effect of monitoring the momentum* by a continuously acting environment (reservoir). The friction coefficient of the oscillator is $\gamma = 2\hbar\kappa/m\omega$.

5 Conclusion

We reviewed here RPI approach to theory of continuous quantum measurements [3] which is especially important since it may be considered as theory of decoherence of open quantum systems in general [7].

Being model-independent, this approach is efficient in various applications (e.g. measurements on harmonic oscillators, visualization of a level transition and measurements of quantum fields). Besides, this approach makes evident fundamental features of the phenomenon of decoherence.

The most important of these features is the dynamical role of information: the evolution of a continuously measured (decohering) system does not depend of details of the measuring medium, but is determined by the measurement readout, i.e. by the information recorded in the environment.

One more remarkable feature is that RPI theory of measurement follows from first principles of quantum mechanics (from Feynman path-integral theory). Therefore, quantum mechanics includes theory of measurements and, contrary to the wide-spread opinion, is conceptually closed. The only unsolved conceptual problems in quantum mechanics are those connected with the role of consciousness in quantum measurements [8].

Presentation of the measured (decohering) open systems by RPI is universal. Although monitoring may be presented simpler, by Schrödinger equation with a complex Hamiltonian, but this is impossible for the measurements which are ‘integral in time’. Even monitoring turns out to be integral in time in the non-Markovian approximation: the number $a(t')$ is actually not the value of A at time t' but (owing to inertial properties of the measuring medium) the average of the values of A over some period containing t' . The weight functional is then more complicated than (8) or (13) [3, Sect. 5.3] and RPI cannot be reduced to a differential equation.

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